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Note

On the number of visibility graphs of simple polygons ☆

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Abstract

We present upper and lower bounds for the number of labelled visibility graphs of simple polygons of n vertices. The lower bound is roughly in the order of $O(n^{2n})$ and the upper bound is $O(n^{3n})$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Research on visibility and illumination problems in combinatorial geometry have motivated the following definition. Given a simple polygon P in the plane with vertices $\{p_1, \dots, p_n\}$, the *visibility graph* $\mathrm{VG}(P)$ has the same vertices as P, and two vertices p_i and p_j are joined by an edge if the segment $p_i p_j$ is contained in P. One also says that p_i and p_j are *visible* inside P. Observe that this definition includes the edges of P as visibility edges. There has been an intense research on visibility graphs in recent years. Two central problems in the area are those of characterizing and recognizing visibility graphs of simple polygons (see [10,11] for surveys). Recently, new interesting approaches have been proposed [13,14].

In this paper we address the problem of counting the number of visibility graphs. More precisely, we want to count the number $g(n)$ of labelled graphs G on n vertices with a distinguished Hamiltonian cycle C, such that $G = \mathrm{VG}(P)$ for some simple polygon P, and such that C corresponds to the boundary of P. Since the visibility graph depends strongly on the shape of the polygon (see for example [9]), it would be clearly quite difficult to compute $g(n)$ exactly, except perhaps for small values of

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n. Thus, we look for asymptotic estimates giving an indication on how $g(n)$ grows with n.

Our results are contained in Theorems 1 and 2. We prove an upper bound of $O(n^{3n})$ on $g(n)$ by considering the relationship between the visibility inside a polygon and the order type of its vertices. On the other hand, we show a lower bound of 'almost' $O(n^{2n})$ by means of a suitable construction. An important point to bear in mind in the asymptotic analysis is that, we only consider terms of order $n^{\alpha n}$ with $\alpha > 0$. This means that we ignore quantities that are only exponential in n.

2. The upper bound

From now on all polygons considered will be simple. Given three points p, q, r in the plane, let $\chi(p, q, r)$ be equal to $+1$ if r is to the left of the directed line pq, zero if the three points are collinear, and -1 otherwise. We recall that the order type of a set $P = \{p_1, \ldots, p_n\}$ of points in the plane is the set of all triples $\chi(p_i, p_j, p_k)$ for i, j, k in $\{1, \ldots, n\}$ (see [8] for a survey on order types). The following lemma is implicit in Welzl's paper [15], and can also be found in [6]. Here we present a short proof for completeness.

Lemma 1. *The order type of the vertices of a polygon determines its visibility graph.*

Proof. Let $P = \{p_1, \ldots, p_n\}$ be a polygon, and assume that we know the order type of the point set P. If p_i and p_j are not visible inside the polygon, it is because the boundary of the polygon crosses the segment $p_i p_j$, and it is clear that we can detect this fact from the order type information.

If the boundary does not cross the segment $p_i p_j$, then p_i and p_j could be *externally* visible instead of *internally* visible. Let q be a point interior to the segment $p_i p_j$, not belonging to any boundary edge; the fact that the visibility between p_i and p_j is internal or external, corresponds to q being interior or exterior to the polygon, respectively. Let r be the ray through p_j with origin at q, and let m be the number of proper crossings between r and the boundary of the polygon; it is well known that the parity of m determines whether q is interior or exterior. Such a point q and the corresponding number m can be easily derived from the order type information, provided one takes care of collinearities (details can be found in the textbook [12]). The proof is now complete. $\square$

From a purely combinatorial point of view, by selecting the value of every triple $\chi(p_i, p_j, p_k)$ independently, there could be in principle as many as $3^{\binom{n}{3}}$ different order types with n points. However, Alon [4] and Goodman and Pollack [7] showed that this is far from being the right answer.

Lemma 2. *The number of* (*labelled*) *order types of n points in the plane is* $O(n^{4n})$, *ignoring terms that grow only exponentially in n.*

The above two results imply a non-trivial upper bound of $O(n^{4n})$ on $g(n)$. The following theorem improves this bound.

Theorem 1. *The number $g(n)$ of visibility graphs of simple polygons of n vertices is* $O(n^{3n})$.

Proof. The order type defined by the vertices of a simple polygon is a rather particular one, since in general the polygon determined by n arbitrary labelled points in the plane will not be simple. Call an order type *polygonal* if the vertices in the order of the labelling define a simple polygon. In view of Lemma 1, any bound on the number of polygonal order types also applies to the number of visibility graphs. We prove next that the number of polygonal order types is $O(n^{3n})$, and the theorem will follow.

We need two simple facts. First, that every order type becomes polygonal after a suitable relabelling (in other words, that every point set admits a simple polygonization). And secondly, that two point sets with the same order type have the same set of simple polygonizations; this is because a polygon is not simple if non-consecutive edges intersect, and this can be detected knowing the order type.

Now let $\mathcal{O}_n$ be the family of all order types of n points, let $\mathcal{P}_n$ be the family of those that are polygonal, and let $\mathcal{S}_n$ be the symmetric group on n letters. Then, we have a natural mapping

$$f\colon \mathcal{P}_n \times \mathcal{S}_m \to \mathcal{O}_n,$$

where f maps a pair (S, σ) into the order type obtained by relabelling the points of the simple order type S according to the permutation σ. For a given order type T, the size of $f^{-1}(T)$ equals the number of simple polygonizations of a set of n points having T as order type. By the result of Ajtai et al. on the number of crossing-free subgraphs of K_n, this number is at most α^n, where α is a constant [3] (see also p. 193 of [2] for an wonderfully simple proof of this fact). By Lemma 2 we have

$$|\mathcal{P}_n| n! \leqslant n^{4n} \alpha^n.$$

Combined with Stirling's approximation for $n!$, we get $|\mathcal{P}_n| \leqslant n^{3n}(\alpha e)^n$, as claimed. □

We close this section with a remark. In [1] it is proved that the set of edges of the visibility graph of a simple n-gon can be covered with cliques and bipartite cliques of total size $O(n \log^3 n)$. This already implies an upper bound of $O(n^{n \log^3 n})$ on $g(n)$.

3. The lower bound

ElGindy showed that every maximal outerplanar graph can be realized as the visibility graph of a simple polygon [5]. The number of maximal outerplanar graphs with n vertices is equal to the number of triangulations of a convex polygon with n sides, and this is well known to be a Catalan number C_n, which is $\Theta(4^n n^{-3/2})$. Hence, this

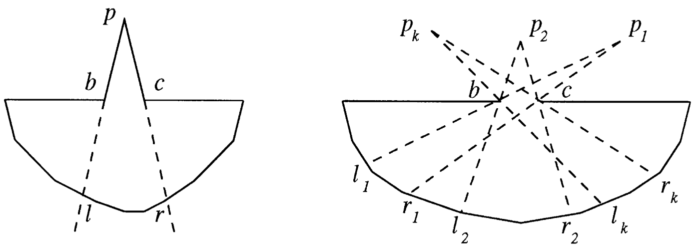

Fig. 1. Visibility through a slot.

gives a lower bound on the number of visibility graphs of order roughly 4^n. This quantity is not a good estimate, since polygons arising in this way correspond to a very restricted class, those admitting a unique triangulation. In this section we prove that $g(n)$ is 'almost' of order n^{2n}, an order of magnitude much closer to the upper bound derived in the previous section.

Theorem 2. *For every* $\varepsilon > 0$, *the number* $g(n)$ *is* $\Omega(n^{(2-\varepsilon)n})$.

Proof. Consider the polygon in Fig. 1a, where the points in the bottom form a convex chain of size m and point p is free to move while maintaining the simplicity of the polygon. We will refer to bc as the *slot* of the configuration. Making the slot narrow enough, we can position p so that it sees any desired interval $\{l, l+1, \ldots, r-1, r\}$ of the convex chain. Since the number of such intervals is equal to $\binom{m+1}{2}$, there are these many different choices of visibility from p.

Next, consider the placement of k points $p_1, \ldots, p_k$, each of them seeing its own interval $[l_i, r_i]$ of the convex chain through the slot bc (see Fig. 1b). The intervals cannot be selected independently, since the resulting polygon might not be simple, but a sufficient condition for simplicity is that

$$l_1 < l_2 < \cdots < l_k \quad \text{and} \quad r_1 < r_2 < \cdots r_k.$$

This is clearly achieved if we select any $2k$ points on the convex chain and label them in increasing order as $l_1, r_1, l_2, r_2, \ldots, l_k, r_k$. As a result we have $\binom{m}{2k}$ choices for the visibility from $p_1, \ldots, p_k$. Note that this quantity is $\Theta(m^{2k})$ for fixed k, and that in the above construction we need $m > k$.

The final step is to place n/k out of a total of n points in the convex chain and distribute the remaining $n - n/k$ points in groups of $k+2$ each, two for making a slot and k as in Fig. 1b. These results in $n(k-1)/k(k+2)$ slots (Fig. 2). By taking the slots narrow enough, we can avoid any conflict that could arise from an interference between different slots, so the selections in each slot are independent. There are $\Theta((n/k)^{2k})$ choices for every slot, and since they are independent the total number is

$$\left[\left(\frac{n}{k}\right)^{2k}\right]^{n(k-1)/k(k+2)}.$$

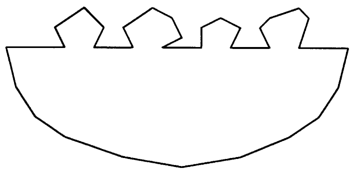

Fig. 2. Visibility through independent slots.

For fixed k, the dominant term in the above expression is $n^{2\frac{k-1}{k+2}n}$. By taking k large enough, we can make it as close as we want to n^{2n}. Hence, the result follows. $\square$

4. Concluding remarks

We have shown that the number of visibility graphs of simple polygons with n vertices is essentially between n^{2n} and n^{3n}. We tend to believe that the true value is closer to the upper bound than to the lower bound, and that a new construction in the spirit of Section 3 could produce a tighter lower bound.

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